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Best L_p Approximate Solutions of Nonlinear Integrodifferential Equations

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In this paper best L_p approximate solutions are shown to exist for a wide class of integrodifferential equations. Using approximation theory techniques, a local existence theorem for solutions is established, and the convergence of the best approximate solutions to a solution is shown.

1. INTRODUCTION

Huffstutler and Stein [8, 9] and Henry [5] have considered best approximate solutions of nonlinear differential equations. Some of these results have been generalized by Kartsatos and Saff [10] and Petsoulas [12] to integrodifferential and integral equations. In each of the references [10, 12], the existence of best approximate solutions is demonstrated only for approximating functions composed of sufficiently many base functions. Computational techniques based on successive approximations have been considered by Olson [11], Henry and Wiggins [6], Allinger and Henry [1], and this author [14]. Since a large number of base functions may make the computations impractical, the focus of this study is the extension of the results of Kartsatos and Saff [10] to any desired number of base functions. The results of Huffstutler and Stein [8, 9] and Henry [5] will thus be generalized to a wide class of integrodifferential equations satisfying continuity conditions.

2. PRELIMINARY DEFINITIONS

Consider the initial value problem (IVP)

$$\begin{aligned} X'(t) &= A(t, X(t)) + \int_0^t F(t, s, X(s))\, ds, \qquad t \in I = [0, \sigma] \\ X(0) &= \Lambda \end{aligned} \tag{1}$$

where $\sigma > 0$.

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Let R denote the set of real numbers and define

$$B = \{(t, s, X) \in I^2 \times R^m : s \leqslant t\}.$$

It is assumed that the functions $A: I \times R^m \to R^m$ and $F: B \to R^m$ are continuous.

For $J = [0, \tau] \subseteq I$ with $\tau > 0$, consider the following norms.

For $U = (u_1, u_2, ..., u_m) \in R^m$, let

$$| U | = \max_{1 \leqslant i \leqslant m} | u_i |$$

and for continuous mappings $U: J \to R^m$ let the L_∞ norm be given by

$$| U |_\infty^J = \max_{t \in J} | U(t)|.$$

We also define the L_p norms

$$| U |_p^J = \left(\int_0^\tau | U(t)|^p \, dt\right)^{1/p}, \qquad 1 \leqslant p < \infty.$$

We next define the approximating set. For $j = 1, 2, ..., m$ let $G_{n,j} = \{ g_{1,j}, g_{2,j}, ..., g_{n,j}\}$, where $g_{i,j} \in C'(I)$ for $i = 1, 2, ..., n$ and for $j = 1, 2, ..., m$. We also assume that for $j = 1, 2, ..., m$ the sets $G_{n,j}$ are each sets of linearly independent functions on any interval $[0, \delta] \subseteq I$, where $\delta > 0$. Let $S_{n,j} = \operatorname{Span} G_{n,j}$ and assume that for $j = 1, 2, ..., m$ the sets $\bigcup_{n=1}^\infty S_{n,j}$ are dense in $C'(I)$ with respect to the norm $\max_{i=0,1} \max_{t \in I} | g^{(i)}(t)|$ for $g \in C'(I)$. We also assume that $g_{1,j}(0) \neq 0$ for $j = 1, 2, ..., m$. Our approximate solutions will be chosen from the set

$$\mathscr{P}_n = \{P \in S_{n,1} \times S_{n,2} \times \cdots \times S_{n,m}: P(0) = A\}.$$

We now introduce the operator

$$L[X](t) = A(t, X(t)) + \int_0^t F(t, s, X(s)) \, ds$$

and set

$$\delta(n, p) = \inf_{Q \in \mathscr{P}_n} | Q' - L[Q]|_p^J .$$

Then $Q_n \in \mathscr{P}_n$ satisfying

$$\delta(n, p) = | Q_n' - L[Q_n]|_p^J \tag{2}$$

will be called a best approximate solution from $\mathscr{P}_n$ to the IVP (1) in the sense of the L_p $(1 \leqslant p \leqslant \infty)$ norm on the interval J. For brevity we will refer to any $Q_n \in \mathscr{P}_n$ satisfying (2) as a BAS of degree at most n on J.

3. Existence of a *BAS* for Each Positive Integer n

THEOREM 1. *Suppose that the functions A and F satisfy the continuity conditions given below* (1). *Then for each positive integer n, for each $J \subsetneq I$ and for $1 \leqslant p \leqslant \infty$ there exists $Q_n \in \mathscr{P}_n$, where Q_n is a BAS of degree at most n on J.*

Proof. The techniques used in the proof of this theorem are quite similar to those given by Coppel [3, p. 17] and adapted by Kartsatos and Saff [10].

Let $1 < p < \infty$ and choose $\{Q_{n,l}\}_{l=1}^{\infty} \subseteq \mathscr{P}_n$ such that $\lim_{l\to\infty} | Q'_{n,l} - L[Q_{n,l}]|_p^J | = \delta(n, p)$.

Set

$$E_{n,l} = Q'_{n,l} - L[Q_{n,l}]. \tag{3}$$

We may assume without loss of generality that

$$| E_{n,l} |_p^J \leqslant \delta(n, p) + 1 \qquad \text{for each } l. \tag{4}$$

Set $M = | A | + \eta(p) + 2$, where $\eta(p) = | A' - L[A]|_p^I = | L[A]|_p^I$. By the continuity of A and F there is a constant $N \geqslant \eta(p) + 1$ satisfying

$$| F(t, s, X)| \leqslant N \qquad \text{and} \qquad | A(t, X)| \leqslant N \tag{5}$$

whenever $(t, s, X) \in \{(t, s, X) \in B: | X | \leqslant M\}$.

Define

$$y(t) = | A | + N(\tfrac{1}{2}t^2 + t) + (\eta(p) + 1)\, t^{1/q}, \tag{6}$$

where $(1/p) + (1/q) = 1$. Since $y(0) = | A | < M$, the number

$$\alpha = \inf(\{t \in J: | y(t)| \geqslant M\} \cup \{\tau\})$$

is positive. Similarly, the numbers

$$\beta_l = \inf(\{t \in J: | Q_{n,l}(t)| \geqslant M\} \cup \{\tau\}),$$

for $l = 1, 2,...,$ are positive.

We next show that $\beta_l \geqslant \alpha$ for $l = 1, 2,...$ and thus establish the fact that

$$| Q_{n,l}(t)| \leqslant M \qquad \text{for} \quad t \in [0, \alpha], \quad 1 < p < \infty, \tag{7}$$

and for each l.

Integrating (3) from 0 to β_l we obtain

$$Q_{n,l}(\beta_l) = \Lambda + \int_0^{\beta_l} L[Q_{n,l}](s)\, ds + \int_0^{\beta_l} E_{n,l}(s)\, ds.$$

Since $s \in [0, \beta_l]$ implies that $| Q_{n,l}(s)| \leqslant M$, the inequalities given in (5) imply that

$$| Q_{n,l}(\beta_l)| \leqslant | \Lambda | + N\beta_l + \tfrac{1}{2}N\beta_l^2 + \int_0^{\beta_l} | E_{n,l}(s)|\, ds.$$

Using Holder's inequality and (4) we have

$$| Q_{n,l}(\beta_l)| \leqslant | \Lambda | + N\beta_l + \tfrac{1}{2}N\beta_l^2 + (\delta(n, p) + 1)\, \beta_l^{1/q}.$$

Since $\delta(n, p) \leqslant \eta(p)$ for $n = 1, 2,...$, we have

$$| Q_{n,l}(\beta_l)| \leqslant y(\beta_l).$$

Since either $\beta_l = \tau$ or $| Q_{n,l}(\beta_l)| = M$ we may conclude that $\beta_l = \tau$ or $y(\beta_l) \geqslant M$. In either case $\beta_l \geqslant \alpha$. Thus (7) is established. The set $\mathscr{P}_n$ is generated by a finite set of base functions which are linearly independent on $[0, \alpha]$, therefore inequality (5) implies that the coefficients of the components of $Q_{n,l}$ are bounded independent of l. Hence there is an element $Q_n \in \mathscr{P}_n$ which is a cluster point of the sequence $\{Q_{n,l}\}_{l=1}^{\infty}$.

From $\lim_{l\to\infty} | E_{n,l} |_p = \delta(n, p)$ it follows that Q_n satisfies (2) and is a BAS of degree at most n. The cases for $p = 1$ and $p = \infty$ require only minor modifications of the above argument and are thus omitted.

4. Local Existence of Solutions of the *IVP* (1)

In this section we consider the IVP (1) on the interval J where we assume Condition S:

$$4N\tau + 3N\tau^2 \leqslant 4.$$

We also make the assumption that for $j = 1, 2,..., m$, $S_{n,j} = \Pi_n$, the set of polynomials of degree at most n.

The techniques used in this section are similar to those used in [6, 7, 14] in other settings.

Set

$$S_n = \{p \in \mathscr{P}_n : | P(t) - \Lambda | \leqslant 2Nt + \tfrac{1}{2}Nt^2 + N\tau t, t \in J\}.$$

Then S_n is a compact subset of $\mathscr{P}_n$. The mapping $T_n : S_n \to \mathscr{P}_n$ is defined as follows.

Let $Q \in S_n$ and suppose $L[Q] = (L[Q]_1, L[Q]_2, ..., L[Q]_m)$. Then define $P = (p_1, p_2, ..., p_m)$ by

$$\max_{t \in J} | p_i'(t) - L[Q]_i(t)| = \inf_{p \in \Pi_{n-1}} | p(t) - L[Q]_i(t)| \tag{8}$$

for $i = 1, 2, ..., m$. Since $P \in \mathscr{P}_n$ implies that $P(0) = \Lambda$ and since continuous functions have unique polynomial best approximations of degree at most n (see [2, p. 80]), P is uniquely defined and we set $T_n[Q] = P$.

THEOREM 2. *If the IVP* (1) *satisfies the continuity conditions given below* (1) *and if condition S is satisfied, then T_n maps S_n into S_n. Furthermore T_n is continuous.*

Proof. We first show that the range of T_n is contained in S_n. Let $Q \in S_n$. Then

$$| Q |_\infty^J \leqslant | \Lambda | + 2N\tau + \tfrac{3}{2}N\tau^2$$

and condition S implies that

$$| Q |_\infty^J \leqslant | \Lambda | + 2 \leqslant M. \tag{9}$$

Set $P = T_n[Q]$ and define

$$E = P' - L[Q]. \tag{10}$$

Since each component of P' satisfies (8), it must be that

$$| E |_\infty^J = | P' - L[Q]|_\infty^J \leqslant | L[Q]|_\infty^J .$$

Integrating (10) using (9) and (5) we obtain

$$\begin{aligned} | P(t) - \Lambda | &\leqslant Nt + \tfrac{1}{2}Nt^2 + \int_0^t | E(s)| \, ds \\ &\leqslant Nt + \tfrac{1}{2}Nt^2 + Nt + N\tau t \\ &\leqslant 2Nt + \tfrac{1}{2}Nt^2 + N\tau t. \end{aligned}$$

Thus $P \in S_n$ and $T_n : S_n \to S_n$.

We next show that T_n is continuous. Let ρ_i be the projection of $\mathscr{P}_n$ onto its ith component.

Then $(d/dt)\, \rho_i \circ T_n[Q]$ is the best polynomial approximation of degree at most n in the scalar L_∞ norm to the function $L[Q]_i$. Since A and F are continuous, the mapping $Q \to (d/dt)\, \rho_i \circ T_n[Q]$ is continuous (see [2, p. 82]). Hence the mapping $Q \to \rho_i \circ T_n[Q]$ is continuous. Recall that $P \in \mathscr{P}_n \Rightarrow P(0) = \Lambda$. Finally, (see [4, p. 101]) we conclude that T_n is continuous.

We may apply the Brouwer fixed point theorem [4] to prove the following theorem.

THEOREM 3. *If the conditions of Theorem 2 are satisfied, then the mapping* $T_n\colon S_n \to S_n$ *has a fixed point* $\bar{Q}_n$ *for each* n.

Define $\bar{E}_n = \bar{Q}_n' - L[\bar{Q}_n]$.

THEOREM 4. *If the conditions of Theorem 2 are satisfied, then* $\lim_{n\to\infty} | \bar{E}_n |_\infty^J = 0$.

Proof. $\bar{E}_n = (e_1, e_2, \ldots, e_m)$ and $\bar{Q}_n = (q_{n,1}, q_{n,2}, \ldots, q_{n,m})$. Then using Jackson's theorem [3, p. 22] we have

$$\max_{t\in J} | e_i(t)| \leqslant \tfrac{1}{6}\omega_i\left(\frac{\tau}{2n}\right) \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

where ω_i is the modulus of continuity of $L[\bar{Q}_n]_i$ for $i = 1, 2, \ldots, m$.

Since $\bar{Q}_n \in S_n$, $| \bar{Q}_n |_\infty^J \leqslant M$ for each n. Also

$$\begin{aligned} | \bar{Q}_n' |_\infty^J &\leqslant | \bar{Q}_n' - L[\bar{Q}_n]|_\infty^J + | L[\bar{Q}_n]|_\infty^J \\ &\leqslant 2 \, | L[\bar{Q}_n]|_\infty^J \end{aligned}$$

and from the bounds on A and F it follows that the sequence $\{\bar{Q}_n\}_{n=1}^\infty$ is equicontinuous. Since F and A are uniformly continuous on compact sets, the sequences $\{L[\bar{Q}_n]_i\}_{n=1}$ for $i = 1, 2, \ldots, m$ are equicontinuous.

Let $\epsilon > 0$. There exists $\delta > 0$ such that if $| t_1 - t_2 | < \delta$, $t_1, t_2 \in J$ then $| L[\bar{Q}_n]_i(t_1) - L[\bar{Q}_n]_i(t_2)| < 6\epsilon$ for $i = 1, 2, \ldots, m$ and for each n. Taking suprema we have

$$\omega_i(\delta) \leqslant 6\epsilon \qquad \text{for} \quad i = 1, 2, \ldots, m.$$

Choose n_0 such that $n \geqslant n_0$ implies that $\tau/2n \leqslant \delta$. Then $n \geqslant n_0$ implies that

$$\max_{t\in J} | e_i(t)| \leqslant \tfrac{1}{6}\omega_i\left(\frac{\tau}{2n}\right) \leqslant \tfrac{1}{6}\omega_i(\delta) \leqslant \epsilon$$

for $i = 1, 2, \ldots, m$. Thus $| \bar{E}_n |_\infty \leqslant \epsilon$ for $n \geqslant n_0$.

We are now ready to state and prove the main theorem of this section.

THEOREM 5. *If condition S and the continuity conditions given under* (1) *are satisfied, then the IVP* (1) *has a solution W.*

Proof. In the proof of Theorem 4 it was shown that the sequence $\{\bar{Q}_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. By Ascoli's theorem, there is a subsequence $\{Q_{n(l)}\}_{l=1}^{\infty}$ and a continuous mapping $W: J \to R^m$, where $\lim_{l\to\infty} | \bar{Q}_{n(l)} - W |_{\infty}^{J} = 0$. Letting $l \to \infty$ in the equation

$$\bar{Q}'_{n(l)} = L[\bar{Q}_{n(l)}] + E_{n(l)}$$

and using Theorem 4 yields the fact that

$$\lim_{l\to\infty} | Q'_{n(l)} - W' |_{\infty}^{J} = 0$$

and that W is a solution of (1) on J.

5. CONVERGENCE RESULTS

In Section 4 it was shown that there exists a sequence $\{\bar{Q}_n\}_{n=1}^{\infty}$ of fixed points of the T_n mappings. This sequence of fixed points contains a subsequence $\{\bar{Q}_{n(l)}\}_{l=1}^{\infty}$ satisfying

$$\lim_{l\to\infty} |\bar{Q}_{n(l)}^{(i)} - W^{(i)} |_{\infty}^{J} = 0, \qquad i = 1, 2,$$

where W is a solution of (1) on J. We now consider the BAS Q_n and the number $\delta(n, p)$ from Section 3. We also once again allow the more general $S_{n,j}$ sets defined in Section 2.

THEOREM 6. *For* $1 \leqslant p \leqslant \infty$, $\lim_{n\to\infty} \delta(n, p) = 0$.

Proof. Since $\bigcup_{n=1}^{\infty} S_{n,j}$ is dense in $C'(J)$ for $j = 1, 2,\ldots, m$, we may choose $P_n \in \mathscr{P}_n$ such that (for a proof of this, see [10])

$$\lim_{n\to\infty} | P_n^{(i)} - W^{(i)} |_{\infty}^{J} = 0 \qquad \text{for} \quad i = 0, 1.$$

The theorem is proved by letting $n \to \infty$ in the inequality

$$\delta(n, p) \leqslant | P'_n - L[P_n]|_p^J .$$

The last theorem of this section will show that the sequence $\{Q_n\}_{n=1}^{\infty}$, where each Q_n is a BAS, has a subsequence which converges uniformly on J to a solution of (1).

THEOREM 7. *Let $1 < p \leqslant \infty$ and suppose A and F satisfy the continuity conditions given below* (1). *If condition S is satisfied, then there is a continuously differentiable mapping $\overline{W}$: $J \to R^m$ and a subsequence $\{Q_{n(l)}\}_{l=1}^{\infty}$ of the sequence $\{Q_n\}_{n=1}^{\infty}$ of BAS's where*

$$\lim_{l\to\infty} | Q_{n(l)} - \overline{W} |_{\infty}^{J} = 0$$

and

$$\lim_{l\to\infty} | Q'_{n(l)} - \overline{W}' |_{p}^{J} = 0.$$

Proof. In the proof of Theorem 1 it was shown that

$$| Q_{n,l}(t)| \leqslant M \qquad \text{for all} \quad t \in [0, \alpha],$$

where M is independent of l and n. Thus

$$| Q_n(t)| \leqslant M \qquad \text{for} \quad t \in [0, \alpha] \text{ and for all } n.$$

We now show that condition S implies that $\alpha = \tau$, If $1 < p < \infty$, then condition S and (6) imply that

$$\begin{aligned} | y(t)| &\leqslant | A | + N(\tfrac{1}{2}\tau^2 + \tau) + (\eta(p) + 1)^{1/q} \\ &\leqslant | A | + 2 - N\tau + (\eta(p) + 1)\, \tau^{1/q}. \end{aligned}$$

If $\tau \geqslant 1$, then $\tau^{1/q} \leqslant \tau$ and

$$| y(t)| \leqslant | A | + 2 + (\eta(p) + 1 - N)\tau.$$

Since $N \geqslant \eta(p) + 1$,

$$| y(t)| \leqslant | A | + 2 \leqslant M \qquad \text{for} \quad t \in [0, \tau].$$

If $\tau < 1$ then $\tau^{1/q} < 1$ and

$$\begin{aligned} | y(t)| &\leqslant | A | + 2 + \eta(p) + 1 - N \\ &\leqslant | A | + 2 \leqslant M \end{aligned}$$

and

$$| y(t)| \leqslant M \qquad \text{for} \quad t \in [0, \tau].$$

Consequently for $1 < p < \infty$, $\alpha = \tau$.

The proof of Theorem 1 was given for $1 < p < \infty$, and it was indicated that only minor modifications were required for the $p = 1$ and $p = \infty$ cases. For the $p = \infty$ case the definition of y given in (6) is slightly changed. The $t^{1/q}$ should be replaced by t. With this observation the above argument also shows that $\alpha = \tau$ for $p = \infty$.

We have thus shown that the proof of Theorem 1 and condition S imply that

$$| Q_n |_\infty^J \leqslant M \qquad \text{for each } n.$$

We next show that $\{Q_n\}_{n=1}^\infty$ is equicontinuous on J. Set

$$E_n = Q_n' - L[Q_n]. \tag{11}$$

Then for $t_1, t_2 \in J$ with $t_1 < t_2$,

$$Q_n(t_2) - Q_n(t_1) = \int_{t_1}^{t_2} L[Q_n](s)\, ds + \int_{t_1}^{t_2} E_n(s)\, ds$$

and

$$| Q_n(t_2) - Q_n(t_1)| \leqslant (N + N\tau)(t_2 - t_1) + \int_{t_1}^{t_2} | E_n(s)|\, ds.$$

Using Holder's inequality, we have

$$| Q_n(t_2) - Q_n(t_1)| \leqslant (N + N\tau)(t_2 - t_1) + \delta(n, p)\, \Delta_p(t_2 - t_1),$$

where

$$\begin{aligned} \Delta_p(t) &= t && \text{if } \ p = \infty \\ &= t^{1/q} && \text{if } \ 1 < p < \infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right). \end{aligned}$$

Thus the sequence $\{Q_n\}_{n=1}^\infty$ is equicontinuous and uniformly bounded on J. By the Ascoli theorem there is a subsequence $\{Q_{n(l)}\}_{l=1}^\infty$ and a continuous mapping $\overline{W}\colon J \to R^m$, where

$$\lim_{l\to\infty} | Q_{n(l)} - \overline{W} |_\infty^J = 0. \tag{12}$$

Integrating (11) from 0 to t we find that

$$Q_{n(l)}(t) = A + \int_0^t L[Q_{n(l)}](s)\, ds + \int_0^t E_{n(l)}(s)\, ds. \tag{13}$$

From Holder's inequality and Theorem 6 it follows that

$$\lim_{n\to\infty} \int_0^t E_n(s)\, ds = 0.$$

Letting $l \to \infty$ in (13) now will yield

$$\overline{W}(t) = A + \int_0^t L[\overline{W}](s)\, ds$$

and consequently $\overline{W}$ has a continuous derivative and is a solution of (1) on J.

Since $\overline{W}' = L[\overline{W}]$ and $Q'_{n(l)} = L[Q_{n(l)}] + E_{n(l)}$ we have that

$$| Q'_{n(l)} - \overline{W}' |_p^J \leqslant | L[Q_{n(l)}] - L[\overline{W}] |_p^J + | E_{n(l)} |_p^J . \tag{14}$$

From (14), (12), and Theorem 6 we obtain

$$\lim_{l\to\infty} | Q'_{n(l)} - \overline{W}' |_p^J = 0.$$

6. Conclusion

Some work has been done on the relationship of the $\overline{Q}_n$ of Section 4 with the Q_n of Section 3. For certain types of differential equations for τ sufficiently small it is shown in [6] that $\overline{Q}_n = Q_n$. This is a topic which needs further study. Iteration methods have been successfully used to compute $\overline{Q}_n$ and numerous examples have been given [1, 6, 7, 11, 14] for a variety of settings.

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